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ELABORATION OF AN ALGORITHM OF DETECTING TESTS' DIMENSIONALITY

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Introduction

Knowledge is a continuously moving concept as once it is mastered it becomes a catalyst for further knowledge discovery (Hampson-Jones, 2011). As lecturers, we are responsible for providing an excellent teaching continuum by well-prepared curriculum plans, sufficient and convenient sources, and well-organized assessment methods to achieve accomplished program outcomes in support of students' overall educational aims. Because assessment is among vital components of effectiveness and adequateness of teaching and learning, the institution should check and support the quality of assessment.

Generally higher education institutions use student questionnaires at the end of the course to evaluate teaching quality. While students' opinions are important and should be included in the assessment of quality, it is obvious that quality of teaching should be explained by different tools based on the meaningful reasons too.

There are various methods to control and enhance the quality. Our proposal is that, there is relation among dimensionality and the quality of an examination and if teaching and assessment is multidimensional and integrated in a course and corresponding examination; such educational process will increase the level and quality of the education. There are several models for the study of dimensionality in the Measurement Theory and instead of general principals of Multidimensional Item Response Theory (MIRT) new approach is introduced and dimensionalities of an actual examination is detected with respect to the prerequisite courses and corresponding examinations.

In this context, first pattern model is designated and it is compared with actual model. Principal components based new methodology is introduced. Convenient acceptance level of the actual model for each of dimensions and the admissibility conditions of the total impact of the internal dimensions of actual exam is determined with regard to reasonable evidences. Quality of a course is interpreted with regard to its dimensionalities.

Structure of the Thesis

Chapter I is literature review. Theory of measurement, basis of measurement on firm mathematical foundation, influences and relations among statistics and measurement theory is given in this chapter. Mathematical determination of scales of measurement and importance of choosing correct scale of measurement in statistical analysis is expressed. Among theories of measurement essential rules of CTT and IRT, MIRT models are presented in this part of the study. Meaning of dimensionality in mentioned three models is determined; factor analysis in IRT and MIRT is shortly presented. Main principles of detecting dimensionalities of a course and using that dimensionality in quality control are given.

In general Chapter II is formal and theoretical definition of the methodology which should be used to detect and determine internal dimensionality of an examination. In this chapter definition of the problem is done, terminology of pattern model and actual model is given, based on the conception of the conditional distribution solution method of the General Multidimensional Problem is determined, brief explanation about multicollinearity is introduced, solution ways of this problem in two directions as least squared method and as conditional expectation is explained with examples. Low rank tensor approximation of the matrix of the independent variables and algorithm of SVD by means of 1-rank tensors approximation are expressed as iterative methods for solution of SVD problem. After that, to describe the internal implicit dimensions and to define impact of prerequisites, SVD and principal components based new approach is theoretically explained in detail. Lastly, how to describe acceptable closeness of expected (pattern) exam results and actual exam results is presented.

In Chapter 3, empirical studies are done with respect to multivariate data analysis procedures and principal components based multidimensional regression analysis procedures. Based on the methodologies given in 2.5.1, 2.5.2, dimensionality detection is done on the strength of prerequisite examinations. Comparison of actual model and pattern model, closeness detection of desirable contributions of internal dimensions of pattern model and estimated contributions of internal dimensions of actual model is done with respects to test of significance. In the comparison of total impact of desirable regression coefficients and total impact of estimated regression coefficients, statistical closeness in the whole is detected by t-test.

Methodology

Based on the multivariate data analysis techniques, MatLab Programming Language and Microsoft Excel Software are used to implement required statistical analysis. Assumptions that grades are distributed according to normal or logistic distributions were used. Mainly three different groups of examinations (actual and prerequisites) are generated such as uncorrelated, low correlated and multicollinear. Each of the examination is simulated for 200 students. According to those examinations different four cases are elaborated. Normality of those prerequisites is presented with their Histograms. Principal Component Analysis (PCA) and Singular Value Decomposition (SVD) methodology are used to define right and left singular vector matrices, singular values and principal components. MatLab is used to demonstrate grades of prerequisites and principal components, and transformation of those prerequisites on principal components. Theory of Conditional Expectation and Multivariate data analysis is used to find estimated regression coefficients of actual examination which are defined here as contributions of internal dimensionalities within actual examination. Multidimensional Regression Analysis is done with regard to actual examination and principal components of prerequisites. Orthogonaility of the principal components is used to avoid the collinearity among prerequisites.

To compare the closeness and acceptance of each of the estimated regression coefficients of actual data and each of the regression coefficients of pattern models the test of significance (t-test) is used. Test of significance is also used to identify the desirable acceptance level of total impact between actual model and pattern model. In the implementation of regression analysis and determination of each of the internal dimensions and total impacts, for the simplicity MatLab code is written and implemented. Outputs of that code are represented as summary statistics within the case studies.

Purpose of the Study

- Determination of conception of internal dimensionalities ;
- Designing of mathematical model of internal dimensions and their impact on an actual examination;
- Theoretical definition and evaluation of coefficients of regression on principal components and estimation of contributions of the internal dimensions factors.
- Estimation of regression coefficients of mathematical model of internal dimensions by Conditional Distribution Methodology and Multivariate Regression Analysis for not-correlated internal dimensions.
- Determining the Principal Component Analysis objects (principal components, Eigen values, transformation matrices) with the methodology of Singular Value Decomposition for strongly correlated.
- Theoretical definition of acceptable closeness of each of the contributions of internal dimensions and total impacts of those contributions between actual model and pattern model.
- Closeness and admissibility comparison of each of the estimated regression coefficients for actual examination and desirable pattern coefficients by test of significance.

- Closeness and admissibility comparison of total impact of estimated internal dimensions and total impact of contributions of internal dimensions of pattern model by test of significance.
- Implementation of iterative method of calculations of left and right matrices of singular vectors and singular values.
- Using MatLab programming language, to write appropriate code for documentation of implementations.

Novelty of the investigation

Although Multidimensional Item Response Theory analyzes dimensionality among different items within a test; in this study, different approach is introduced. Existence of internal (implicit) dimensionalities in an examination is searched on the strength of the ability levels of several congener courses' examinations which are named here as grades of prerequisite examinations. Using those prerequisite examinations dimensionality of actual examination is detected and described to represent connection among current course and prerequisite courses. For that reason principal components based new methodology is used. Eventually, adequacy and quality of the actual course and actual examination is detected with respect to the elaboration of actual examination's dimensionality.

Scientific and Practical Importance

In the thesis study, to compute regression coefficients:

1. Calculation of singular vector and singular values of prerequisite scores is done;
2. Using formulas $\Sigma_{12}\Sigma_{22}^{-1} = \begin{bmatrix} \frac{(y,U_1)}{\lambda_1} & \frac{(y,U_2)}{\lambda_2} & \dots & \frac{(y,U_k)}{\lambda_k} \end{bmatrix} = \beta^T$ and $\alpha = V\beta$, those regression coefficients are determined.

While the prerequisite examinations are un-correlated, low-correlated, or nearly perfect collinear, this approach (together with the algorithm of SVD by Means of Rank Tensors Approximation (see 2.5.3.2)) gives possibility to avoid calculation of inverse covariance matrix, procedure of least square methods and simply allows obtaining for sought regression coefficients directly by means of $\alpha = V\beta$.

In the multicollinear case to avoid high collinearity standard regression procedure is implemented on the orthogonal principal components. This regression gives β coefficients which can be transformed into α by means of $\alpha = V\beta$. Those estimated α coefficients are used as contributions of the dimensionalities of actual examination. Therefore dimensionality detection is done with regard to prerequisite courses and corresponding examinations.

Structure and volume of the work

The thesis study is 134 pages and consists of 3 chapters, a list of references and list of figures and list of tables.

Definition of the Problem

Let's call the set of requirements defined by Educational Institution the exam's Pattern Model, and then the results of the real (actual) exam represent Actual Model. It leads to the definition of the following problem: estimation of the Closeness of the Actual Model to the Pattern one.

Now we have to express both models in mathematical form. With this in view, let's introduce parameters for pattern model:

$\boldsymbol{\alpha^0}$ is a desirable share of M, which is defined by prerequisites (it can be, for example,30% of M);

α_i^0 (i=1, 2… k) share of each of k prerequisites;

Let's denote the sum of prerequisites share as

$$\alpha^0 = \sum_{i=1}^{k} \alpha_i^0 \ . \tag{1}$$

The meaning of the restriction (1) is easily understandable: total impact of prerequisites or internal dimensions of the actual exam should be equal to the simple sum of separate dimensions.

The above formula means that the sum of prerequisites defines α^0 % of the possible score of the actual exam under consideration.

And it is clear that,

$(1 - \sum_{i=1}^{k} \alpha_i^0)$ is share of the possible scores which is defined by knowledge of actual course under consideration.

Thus the pattern model is represented by means of a set of values of α_i^0 $while$ $(i = 1,2,\dots,k)$ and (1), where k is the number of prerequisites.

The results of the sample exam can be represented as the following linear model

$y_i = \sum_{i=1}^{k} \alpha_i x_{ij} + \varepsilon_t$ (i=1, 2… n) or in the matrix notation

$$y = X\alpha + \varepsilon, \tag{2}$$

where $\boldsymbol{y}$ is the vector of observed scores times α, i.e.$z_i = \alpha^0\, y_i$ gives share of the score y_i, i.e. the part of the total score, and thus $(1-\alpha^0)y_i$ is share which is defined by ability in the discipline that is not taken into consideration by prerequisites,

Thus, having the matrix of the prerequisites results, one has pattern model parameters α_i^0 $(i = 1,2 \dots k)$ and then compare its closeness to the actual model parameter α_i (i=1, 2… k). Closeness of the two vectors should be considered as positive evaluation of the exam's results, because the internal dimensionalities structure satisfies quality requirements expressed by the pattern model.

The above creates the problem of determination of natural restriction imposed on estimated coefficients:

$$\sum_{i=1}^{k} \alpha_i \approx \alpha^0. \tag{3}$$

Meaning of (3) is that the sum of estimated coefficients should be close to α^0, whereas each of α_i can be different from α_i^0 . We introduce two notions:

1. Closeness per each dimension,
2. Closeness in the whole.

We have to note the testing of the closeness in the whole is equivalence of testing of holding of restriction (3).

Evaluation of the parameters of the model (2) is a problem of estimation of multidimensional linear regression parameters. We have to discuss two cases of the approaches of the model:

1. vectors of the matrix x_{ij} (scores of prerequisites) are non-correlated or low-correlated: it means, that their covariance matrix is not singular and its inverse matrix exists;
2. vectors of the matrix x_{ij} (scores of prerequisites) are strongly correlated : it means, that their covariance matrix is singular and its inverse matrix either does not exists or its determinant is close to zero, so that the matrix is bad-conditioned; It is clear that these cases require different approaches.

Determining Internal Dimensions and Defining Impact of Prerequisites

Principal Components Method

The singular value decomposition (Hardle, W.; Simar, L., 2003) is one of the most important tools in linear algebra (Ipsen, 2009). In the singular value decomposition it is known that any matrix

$X \in \mathbb{C}^{nxk}$ can be represented as the product of three matrices. These are left singular vector matrix $U \in \mathbb{C}^{nxn}$, singular values L which can be shown as a diagonal matrix with non-negative entries, and transpose of right singular vector $V \in \mathbb{C}^{kxk}$. The singular value decomposition of X is given by:

$$\mathrm{X} = ULV^T, \tag{4}$$

where U and V are unitary matrices (reference), the columns of U is orthonormal eigenvectors of XX^T, and the columns of V are orthonormal eigenvectors of X^TX, and L is a diagonal matrix containing the *square roots of eigenvalues* from U or V in descending order (Baker, 2013).

"A basis $\{v_1, v_2, .., v_n\} \in E^n$ is orthonormal if,

1. $v_i v_j = 0$ (whenever $i \neq j$ the vectors are mutually orthogonal), and
2. $v_i v_i = 1$ (they are all unitary matrices) (Lerner, 2008)".

A set of vectors is said to be orthonormal if the norm of each vector in the set is equal to one. Since U and V are orthonormal, their norm is

$\|U\| = \|V\| = 1$. Because U and V are unitary matrices,

$UU^T = I \quad and \quad VV^T = I.$

There is obvious relation between SVD and PCA which is shortly explained below:

Simply the PCA requires the computation of the eigenvalues and eigenvectors of the covariance matrix of X, which is the product $\mathbf{XX^T}$. Because the covariance matrix is square and symmetric matrix and eigenvectors can be normalized such that they are orthonormal, where $X = ULV^T$; SVD is applicable to that covariance in the following way:

$$XX^T = (ULV^T)(ULV^T)^T,$$

$$XX^T = (ULV^T)(VLU^T), \qquad \text{since } V^TV = I$$

$$XX^T = UL^2U^T. \tag{5}$$

where L^2 is a diagonal matrix whose diagonal elements are the eigenvalues ($\lambda_1^2, \lambda_2^2, \dots, \lambda_k^2$) of XX^T. Here the connection can be seen easily: The square roots of the eigenvalues of XX^T are singular values of X (Grodner & Grove, 2007). This connection helps to use PCA and SVD for the purpose of the study within convention.

If we multiply both sides of the equation 19 by right singular vector *V*, equation becomes $XV = ULV^TV$. In that case we can compute the principal components based on multiplying the left singular vector U and diagonal matrix of singular values L or based on multiplication of matrix X and right singular vector matrix V.

$XV = UL.$ (6)

Both XV and UL are principal components which can be denoted as "P" and if we substitute X by singular value decomposition of X,

$y = X\alpha + \varepsilon$, would be

$y = ULV^T\alpha + \varepsilon$, and since ***UL=P***, "y" can be represented as

$y = PV^T\alpha + \varepsilon.$ (7)

If we denote $\boldsymbol{V^T\alpha}$ as $\boldsymbol{\beta}$,

$y = P\beta + \varepsilon.$ (8)

If we can find such $\boldsymbol{\beta}$ then $\boldsymbol{\alpha}$ can be solved from

$V^T\alpha = \beta.$ (9)

Regression coefficients' vector $\boldsymbol{\alpha}$ can be found using principal components of X which are denoted as P**.** It is clear that first $\boldsymbol{\beta}$ should be calculated.

Theoretical Evaluation of Regression Coefficients on Principal Components and Contributions of the Inner Dimensions

After estimating of $\boldsymbol{\beta}$ one can find $\boldsymbol{\alpha}$ from the equation 9**.** Regression coefficients' vector $\boldsymbol{\alpha}$ can be found by means of usage of the methodology of the Regression on Principal Components (PC)**.**

Definition 2.1: Let Y denote the (n×k) data matrix, where n≥k. Any symmetric matrix (k×k) S whose (i, j) element measures the degree of association between variables Y_i and Y_j is known as association matrix. There are four types of Gramian[1] Association Matrices (Gentle, 2007)used in statistics: Inner Product Matrix, Cosine matrix, Covariance matrix, Correlation matrix.

All four types of matrices have common nature, but also some different characteristics. We do not consider them in more detail, but only underline that they, being used in regression analysis, provide the same final results (from computational point of view). Hereafter we'll use mostly Inner Product Matrix instead of covariance one. Note that the matrix of PC contains only k columns, where number of k is equal to the number of nonzero singular values of the matrix L. So using singular values and first k columns of U, it is possible to indicate nxk matrix P as decomposition of column vectors which is given below:

$P = [\lambda_1 U_1 \quad \lambda_2 U_2 \quad \lambda_3 U_3 \ldots \lambda_k U_k]$, or

$P = [P_1 \ \ P_2 \ \ P_3 \ldots P_k]$, where P_i is i^{th} Principal Component.

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[1] The Gramian matrix is an Hermitian matrix of inner products of a set of vectors $v_1, \ldots, v_k$.

The purpose was using principal components finding β coefficients, and after that determining regression coefficients $\boldsymbol{\alpha}$ to identify impact of internal dimensions. For that reason, let's consider the Inner Product Matrix (IPM) of the system of vectors $y,\ P_1,\ P_2,\ P_3 \dots P_k$,

Then the IPM of the system of vectors is

$$\Sigma = \begin{bmatrix} (y,y) & (y,P_1) & \cdots & (y,P_k) \\ (P_1,y) & (P_1,P_1) & \dots & (P_1,P_k) \\ \vdots & \vdots & \ddots & \vdots \\ (P_k,y) & (P_k,P_1) & \cdots & (P_k,P_k) \end{bmatrix}.$$

It is the very well-known fact that, for the multivariate normally distributed data, β, being the conditional expectation of the random vector y for certain values of independent variables P_i (i=1,2,…,k), can be represented as, $\beta = \Sigma_{12}\,\Sigma_{22}^{-1}$, where

$$\Sigma = \left[\begin{array}{c|ccc} (y,y) & (y,P_1) & \cdots & (y,P_k) \\ \hline (P_1,y) & (P_1,P_1) & \dots & (P_1,P_k) \\ \vdots & \vdots & \ddots & \vdots \\ (P_k,y) & (P_k,P_1) & \cdots & (P_k,P_k) \end{array}\right] \qquad \overbrace{}^{\Sigma_{12}} \quad \underbrace{}_{\Sigma_{22}}$$

Principal components are geometrically "orthogonal" to each other since they are projection of initial entries on the orthogonal singular vectors. Thus, it is clear that within sub-matrix Σ_{22} except diagonal elements all remainder dot products are zero, since P_i and P_j are orthogonal. Furthermore since singular vectors are orthonormal, $\|U_i\| = 1$,

$(P_i, P_i) = (\lambda_i U_i, \lambda_i U_i) = \|\lambda_i\ U_i\|\|\lambda_i U_i\| = \lambda_i^2 \|U_i\|\|U_i\| = \lambda_i^2.$

Therefore,

$$\left(P_i, P_j\right) = 0 \quad while\ i \neq j\ and\ \left(P_i, P_j\right) = \lambda_i^2 \quad while\ i = j. \tag{10}$$

And therefore Σ_{22} can be represented as dot product of $L^T\ and\ L$. Thus,

$$\Sigma_{22}^{-1} = \begin{bmatrix} 1/\lambda_1^2 & 0 & \dots & 0 \\ 0 & 1/\lambda_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1/\lambda_k^2 \end{bmatrix}. \tag{11}$$

To calculate the vector of inner products among observed scores and principal components which are represented as Σ_{12}, we follow the next inner product

$(y, P_i) = (y, \lambda_i U_i) = \lambda_i (y, U_i)$. (12)

As a result Σ_{12} is 1xk row vector and Σ_{22}^{-1} is kxk square matrix, regression correlation coefficient β would be 1xk row vector

$$\Sigma_{12}\Sigma_{22}^{-1} = \left[\frac{(y,U_1)}{\lambda_1} \quad \frac{(y,U_2)}{\lambda_2} \quad \dots \quad \frac{(y,U_k)}{\lambda_k}\right] = \beta^T . \quad (13)$$

Now we can estimate $\boldsymbol{\alpha}$ with respect to $\boldsymbol{\beta}$**.**

$VV^T\alpha = V\beta$ and since $VV^T = I$

$$\alpha = V\beta. \quad (14)$$

Thus, the above represented approach consists of two steps:

3. Calculation of singular vector and singular values of prerequisite scores;
4. Using formulas (13) and (14) to compute regression coefficients.

This approach (together with the algorithm of SVD by Means of Rank Tensors Approximation) gives possibility to avoid calculation of inverse covariance matrix, procedure of least square methods and simply allows obtaining for sought regression coefficients directly by means of the formula (14).

In the case of multicollinearity we use to calculate for sought coefficients, instead of analytical expressions (13) and (14), standard regression procedure on principal components. This regression gives β coefficients which can be transformed into α by means of (14).

Theoretical Definition of Acceptable Closeness of Each of the Contributions of Internal (Implicit) Dimensions and total Impacts of those Contributions between Actual Model and Pattern Model

Observed grades can be represented as the following linear regression model

$y = X\alpha + \varepsilon$, where alpha values are estimated values; ε is a vector of normally distributed mutually independent random values with zero means.

Predicted grades can be denoted as $\hat{y} = X\alpha$ and (15)

pattern exam grades can be represented as $y_t = X\alpha^0$, (16)

where α^0 is desirable (pattern) contributions of inner dimensions defined by prerequisites.

To test acceptance of the exam under consideration one has to estimate closeness of α and α^0 vectors. Because both α and α^0 are k-dimensional vectors, we have to test two cases:

1. Statistical closeness of each coordinates (per each of dimensions) of the vectors α and α^0, that is

$H_o: \alpha_i = \alpha_i^0 \quad (i = 1,2 \dots, k),$

$H_1: \alpha_i \neq \alpha_i^0 \quad (i = 1,2 \dots, k),$ (17)

and

2. Closeness of the sums of components of vectors of α and α^0 that is closeness in the whole.

$H_o: \sum_{i=1}^{k} \alpha_i = \sum_{i=1}^{k} \alpha_i^0$,

$H_1: \sum_{i=1}^{k} \alpha_i \neq \sum_{i=1}^{k} \alpha_i^0.$ (18)

1. Statistical closeness per each of dimensions.

For the comparison of closeness of α and α^0 , and for the verification of truth or falsity of a null hypothesis H_o, test of significance (Gujarati, 2004; Rencher, 2002), t-test, is used. Taking into account that the distribution of the sum of two (or more) normally distributed random variables is again normal distribution with expectation and variance equal to sums of expectations and variances; in this context, the test of significance will be

$$t_{i,N-K} = \frac{(\alpha_i - \alpha_i^0)}{\sqrt{S^2\, Q_{ii}^{-1}}}, \qquad (19)$$

Where,

N: number of observations, N-K: degrees of freedom, S: total error of regression model,

Q: covariance matrix of independent variables.

If $\text{SSR} = \sum_{i=1}^{N} (y_i - \hat{y}_i)^2$ is sum of squared errors then (20)

$$S = \sqrt{\frac{SSR}{N-K}} = \sqrt{\frac{\sum_{i=1}^{N}(y_i - \hat{y}_i)^2}{N-K}}$$ (Standard deviation of regression coefficient α_i .) (21)

We can reject the null hypothesis, if $|t_{i,N-K}| \geq t_{\alpha/2,N-K}$, (22)

where confidence level is $\alpha = 0.05$, and $t_{\alpha/2,N-K}$ is a critical value from the t-table. In this case our statistic is said to be statistically significant since t-value lies in the critical region. We can find α_i such that,

$\alpha_i = (V\beta)_i, \quad 1 \leq i \leq k$. According to null hypothesis (17)

$H_o: \alpha_i^0 - (V\beta)_i = 0$, and $H_1: \alpha_i^0 - (V\beta)_i \neq 0.$

If calculated t value falls in the acceptance region, we can assume that differences between α_i and α_i^0 is because of randomness. And it is possible to assume that

$\alpha_i - \alpha_i^0 \approx 0.$

2. Statistical closeness in the whole.

To determine criterion for the purpose we have to compare the distribution of the sum of k normally distributed random variables α_i $(i = 1,2,..,k)$ with nonrandom constant $\sum_{i=1}^{k} \alpha_i^0 = \alpha^0$. Similar the previous case and again taking into account that the distribution of the sum of k normally distributed random variable is again normal distribution with expectation and variance equal to sums of expectations and variances. In that case t-test, will be

$$t_{i,N-K} = \frac{(\sum_{i=1}^{k} \alpha_i - \sum_{i=1}^{k} \alpha_i^0)}{ST}, \text{ where } ST = \sqrt{\frac{S^2}{\sum_{i=1}^{k} Q_{ii}}}. \quad (23)$$

If calculated t value falls in the acceptance region, we can assume that differences between $\sum_{i=1}^{k} \alpha_i$ and $\sum_{i=1}^{k} \alpha_i^0$ is because of randomness. And it is possible to accept the null hypothesis $H_o: \sum_{i=1}^{k} \alpha_i = \sum_{i=1}^{k} \alpha_i^0$.

In the other case, large t value is evidence against the null hypothesis and there should be a systematic error since estimated t is out of the acceptance region. Hence,

$$H_1: \sum_{i=1}^{k} \alpha_i \neq \sum_{i=1}^{k} \alpha_i^0. \quad (24)$$

Numerical Example: Determining Internal Dimensions and Defining Impact of Prerequisites by Principal Components Method

In the implementation part of the thesis different 12 case studies are done. One of them represented below to show a sample application. This is chosen from the section 3.3. Prerequisites are low-correlated. After the simulations study 200x3 matrix A is determined as combination of dependent variable vector “Y” and column vectors of two prerequisites. Dependent variable Y is considered as α % exam under interest. Y is named as actual exam and vectors of independent variables X_1, X_2 are considered as prerequisite 1 and prerequisite 2 respectively. $A = [Y \quad X_1 \quad X_2]$. X_1, X_2 are normally distributed and normality of their distribution is depicted by histograms below.

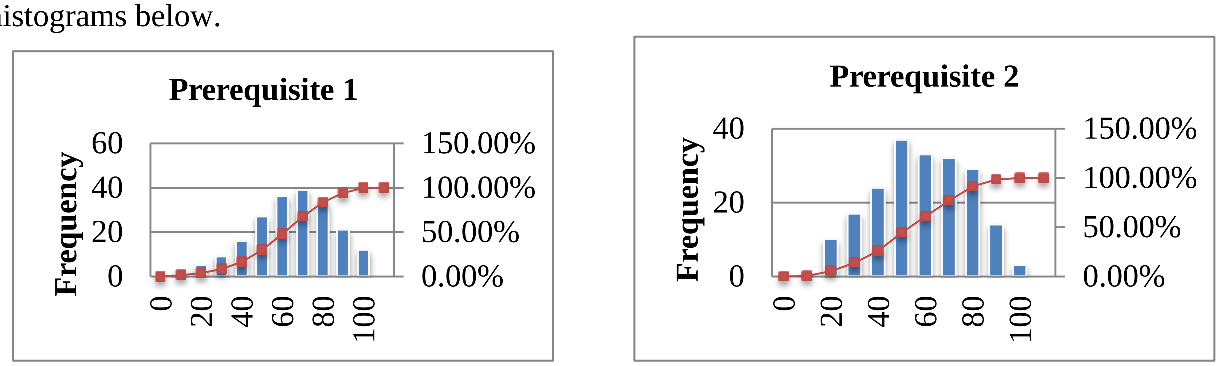

Figure 10: *Normally distributed prerequisites 1and prerequisite 2, illustrated by histogram.*

Low correlation between these two prerequisites is illustrated below by scatter plot. Correlation between them is 0.20.

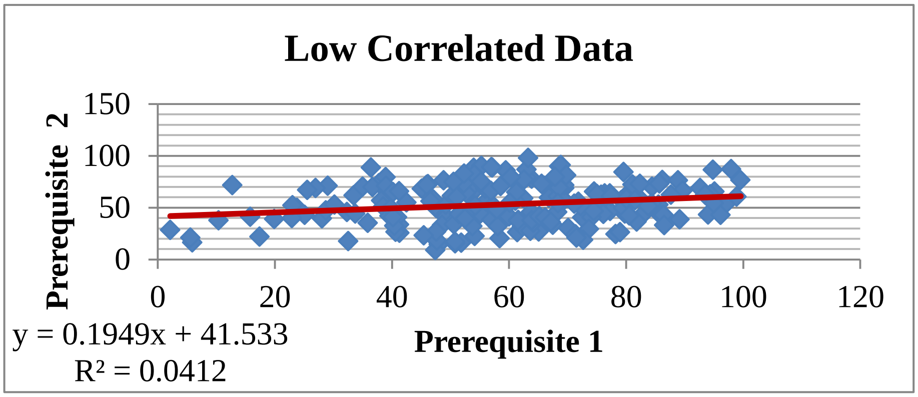

Figure 12: *Scatter plot of the low correlated two prerequisites.*

Desirable contributions of internal dimensions of pattern model are defined as $\alpha_1^0 = 0.15$, $\alpha_2^0 = 0.2$ for the prerequisite 1 and prerequisite 2. Desirable total impact in actual exam is: $\alpha^0 = \alpha_1^0 + \alpha_2^0 = 0.35$. If the actual examination is three dimensional, two of them should be related by the examinations of two prerequisites and one dimension with the actual exam itself. For the detection of implicit dimensions regression coefficients would be described by $\boldsymbol{\beta} = \boldsymbol{\Sigma_{12}\Sigma_{22}^{-1}}$.

Instead of direct usage of prerequisites principal components are used in this study. First to avoid the collinearity second for the simplicity of the evaluation of regression coefficients, principal components based multidimensional regression analysis is done. Using the fundamental principles of conditional distribution, contributions of internal dimensions are determined.

Instead of matrix A the following matrix B is taken into the consideration which is defined by actual exam Y and two principal components:

$B = [Y \quad PC_1 \quad PC_2]$.

Using SVD methodology, matrices of right and left singular vectors are calculated and determined as V and U. To obtain principal components of independent variables (prerequisites) dot product of U and matrix of singular values L is used (P=U*L). It is possible to use X*V to find same principal components.

When we follow the theoretical procedures of section 2.5.2 it is necessary to describe Σ_{12} and Σ_{22} to find mentioned regression coefficients. In that part, Σ_{22} is simply determined as inner product of L^T *and* L, where L is the vector of singular values. Diagonal elements of L are 1181.623 and 259.66. It is possible to evaluate Σ_{22} and Σ_{12} from the Inner Product Matrix (IPM).

IPM of the system of vectors of actual exam grades and principal components is used to define the contributions of the implicit dimensions. IPM is:

$$\Sigma_{IP} = \begin{bmatrix} 140631.832564039 & -442445.084863982 & -3262.93386929556 \\ -442445.084863982 & 1396231.61327054 & 0 \\ -3262.93386929556 & 0 & 67426.5997287211 \end{bmatrix}$$

When the inner product of dependent variable y (actual exam) and principal components of prerequisites is determined in 2.5.2 as

$(y, P_i) = (y, \lambda_i U_i) = \lambda_i (y, U_i)$,

Σ_{12} can be found in MatLab as dot product of y and principal components by:

$\Sigma_{12} = Y^T * U * L$. Therefore Σ_{12} is:

$\Sigma_{12} = [-442445.084863982 \quad -3262.93386929556]$, and

$\Sigma_{22} = \begin{bmatrix} 1396231.61327054 & 0 \\ 0 & 67426.5997287211 \end{bmatrix}$. Inverse of Σ_{22} is

$\Sigma_{22}^{-1} = \begin{bmatrix} 7.16213549740216e-07 & 0 \\ 0 & 1.48309421507732e-05 \end{bmatrix}$.

Hence, $\Sigma_{12}\Sigma_{22}^{-1}$ gives β^T and regression coefficients of principal components are:

$\beta = \begin{bmatrix} -0.3169 \\ -0.0484 \end{bmatrix}$.

Using actual exam grades and those principal components with help of described model in section 2.5.2, the following right singular vector matrix V is found:

$V = \begin{bmatrix} -0.742408852018603 & 0.669947084809256 \\ 0.669947084809256 & -0.742408852018603 \end{bmatrix}$.

Using $V * \beta$, estimated implicit dimensions are:

$\alpha = \begin{bmatrix} 0.2028 \\ 0.2482 \end{bmatrix}$.

To compare the admissibility of these coefficients t-statistics is used and that comparison is made by the following code:

```
CV=cov(X1);
t=(ALET-ALFA).*sqrt(diag(CV)/er);
```

As it is shown above beside the error term, to represent the closeness of both pattern and actual model the following covariance matrix of prerequisites is also necessary:

$\Sigma_X = \begin{bmatrix} 423.363867930915 & 78.4741277061488 \\ 78.4741277061488 & 411.615452512560 \end{bmatrix}$.

While covariance matrix is the above Σ_X and mean squared error is 1.3686, t-values are found as 0.9211 and 0.8291 respectively with regard to estimated and predefined alpha coefficients. According to the t-critical value 1.6526 those t-values show that both impacts of estimated implicit dimensions are acceptable for degrees of freedom 197. Because, the t-values do not exceed the t-critical value at the significant level of .05, therefore, we fail to reject the null hypothesis which is the difference between estimated and desirable regression coefficients is zero. Additionally, when we compare the closeness of the *total impacts* of both estimated and desirable inner dimensions, the below given MatLab code is used:

```
d=sum(ALFA)-sum(ALET);
sigmat=sqrt(sum(er./diag(CV)));
t=d/sigmat;
```

In the comparison of the total impact of the contributions of the inner dimensions among the desirable model and in the actual model, t-value is found as -1.2371. This value again indicates that since absolute value of -1.2371 does not exceeds the t-critical value the closeness is reasonable. That is the indicator of acceptable total impact of actual model. The actual examination has satisfactory relation with pattern model's prerequisites. In this manner, it is indicated that, actual model does not consist and measure only one of ability and it does not concentrated on one dimension. Actual model on the strength of two prerequisites and the current course itself is three dimensional. Therefore, a course is elaborated from the dimensionality and at the same time quality point of view, thus current state proves that implemented course has adequate quality with regard to the appropriate utilization of broad perspective of background knowledge.

Undesirable Actual Model: One of the Estimated Inner Dimensions and Total Impact of Both Inner Dimensions Exceeds "a Desirable Coefficient and Desirable Total Impact" Respectively

In the education process different situations can be faced when the focus is the quality of a course. Beside acceptable, adequate actual courses, insufficient implementation of courses also can be find place in the education cycle. Different scenarios are taken into the consideration with respect to the experts' opinions, to represent effectiveness of our proposals and here undesirable (unqualified) case is researched. Desired impacts for a well-organized, satisfactory actual course are considered as $\alpha_1^0 = 0.15;\ \alpha_2^0 = 0.20$.

As it was mentioned in our study regression is done on the principal components. Right singular vector matrix which is found by singular value decomposition is shown below

$$V = \begin{bmatrix} -0.742408852018603 & 0.669947084809256 \\ -0.669947084809256 & -0.742408852018603 \end{bmatrix},$$

$\beta = \begin{bmatrix} -0.3833 \\ -0.1226 \end{bmatrix}$, and since $\alpha = V\beta$,

In the actual examination, estimated first regression coefficient is found “$\alpha_1 = 0.2024$” and with respect to the first t value “0.9211”, share of the prerequisite 1 is acceptable. However, according to the second internal estimated dimension $\alpha_2 = 0.3478$, t-value “2.5633” is larger than t-critical value “1.6526” for the degrees of freedom197, thus situation cannot be explained as randomness, and contribution of the prerequisite 2 is not acceptable so, our null hypothesis should be rejected.

While total impact of the two prerequisites is 0.35 in the pattern model, total impact of estimated coefficients of inner dimensions in the actual model is 0.5502 and these results are very far from each other. Comparison of deviation of the total impact (sum of inner contribution of pattern model) from the inner contribution of estimated model is done with respect to the covariance of the pattern model, error between actual exam grades and desirable pattern grades for degrees of freedom 197 and t-value is found as -2.4720 and it is greater than t-criterion value “1.6526”, therefore, such actual examination cannot be considered as acceptable model on the strength of the total impacts of estimated coefficients and pattern coefficients too. It is obvious that examination is three dimensional (prerequisite 1, prerequisite 2 and actual examination) but the problem is that distribution of previous courses are highly represented in the actual exam.

This situation affects the quality of the actual course. Because almost 35% of the second prerequisite allows explaining the current exam, this case can be considered as the indicator of the lack of the transformation of new knowledge which should be given in the current semester. It may be interpreted that actual examination significantly depends on the old taught knowledge in the previous courses since total sum of implicit coefficients are approximately 0.55. This case depicts that new course has high relation with background knowledge. From the quality point of view a course should increase the capacity of students; however it can be interpreted that given sample course does not satisfy such condition adequately.

Undesirable Contribution of one of Inner Dimensions but Acceptable Total Impact

In this part the purpose is to demonstrate different scenario to show the validity of our theoretical approach. For that reason new data is designated on the strength of the contribution of the estimated new coefficients. Simply when we directly analyze that data in the MatLab code we see that for the actual exam estimated inner dimensions are 0.2524 and 0.1478 respectively. At the same time contribution of the inner dimensions is given as 0.1500 and 0.2000 for the pattern model. Based on the first prerequisite t-value is determined as 1.7985 and statistically it is significant that the null hypothesis should be rejected. This situation shows that there is difference between contribution of the estimated inner dimensions of the first prerequisite and coefficient of the pattern model's first prerequisite which cannot be describe because of randomness. So, the first estimated contribution is not agreeable.

With regard to contribution of the second inner dimension, t-value is obtained as -0.9042 this is the case that allows acceptance of the contribution of the estimated second prerequisite. Besides, while the total impact of the estimated coefficients is approximately 40% this impact in the actual exam is acceptable because t-value is -0.6196. In fact the desirable total impact α^0 was 35%. Algebraically there is not so big difference (only 5%), and for the share of prerequisites in the actual exam this difference can be acceptable with respect to t-value. As a conclusion it can be interpreted that the actual exam is three dimensional and it almost satisfies the assumptions of the qualified exam.

Undesirable Contribution of Inner Dimensions and Unacceptable Total Impact

In the previous three cases different possibilities are discussed. Additionally very important a new scenario is elaborated in the fourth case. According to the new data the following results of the actual exam is obtained.

Desirable pattern model coefficients are described as $\alpha_1^0 = 0.15$; $\alpha_2^0 = 0.20$. Closeness of those pattern parameters and estimated contributions are again detected with respect to t-values. T-values are found as -1.7172 and -2.1192 with regard to prerequisite 1 and prerequisite 2 respectively. Both of them are greater than t-critical value and statistically both contributions of estimated inner dimensions are not acceptable, null hypothesis should be rejected. There is distinctive difference between expected inner contributions α_1^0, α_2^0 and estimated coefficients $\alpha_1 = 0.0524$ and $\alpha_2 = 0.0778$ respectively. Meaning of those estimated regression coefficients is that effects of both prerequisites are very weakly observed in the actual exam.

As it was mentioned previously both prerequisites and actual exam itself are based on the similar and parallel abilities. They are considered in the same frame work of courses. In this sense, relation between the courses is obvious from the pedagogical point of view. *According to the assumptions of Item Response Theory ability of a student is invariant for the particular time.* In general, "if the conditions of the exams and level of teaching is reliable" when a student is successful/unsuccessful in a course, the same student should be again successful/unsuccessful in the similar and connected topics or corresponding new course. This approach can be generalized to a group of students too. Thereby, relation between current course and background courses which are named here prerequisites can be analyzed based on the exam results of those courses. Usage of pre-gained knowledge is important to increase the level of a course because new knowledge and new talents are constructed on those bases. Inadequate level of education process does not consist of those bases. Reflection of prerequisites should be observed in a good quality exam.

In this context, while in a qualified course the expected impact of a prerequisite is determined 15% ($\alpha_1^0 = 0.15$), if contribution of a prerequisite is very low (almost 5%) like faced above ($\alpha_1 = 0.0524$), that situation can be interpreted as *lack of usage of pre-knowledge or lack of connection among pre-gained and current knowledge.*

It is given that $\alpha_2^0 = 0.20$ but estimated contribution is $\alpha_2 = 0.0778$ for prerequisite 2. According to those coefficients, t-values are getting as -1.7172 and -2.1192, and both exceed t-critical value (1.6526). Hence, impact of the two of the estimated inner dimensions for the actual exam does not acceptable. In the current course relation between background information and new one is not observed sufficiently. Furthermore, total impact of those estimated contributions of inner dimensions is approximately 13% and expected total impact was determined as 35%. Closeness of those total impacts using t-test described. Eventually, actual model and pattern model are very far from each other from the total impact analysis point of view with regard to t-value (2.7146). Thus, actual model is not acceptable and allowable. The actual course highly concentrated on one dimension, course itself.

Conclusions

A course and corresponding actual examination is elaborated from the dimensionality and at the same time from the quality point of view. Mainly uncorrelated, low correlated and multicollinear three different cases are generated and elaborated for detecting dimensionality of an examination. In detail four distinct situations are taken into consideration, and consequently following facts are observed and interpreted:

Case1: Desirable closeness of actual model and pattern model,

Case 2: Undesirable actual model while one of the estimated inner dimensions and total impact of both inner dimensions exceeds "a desirable coefficient and desirable total impact",

Case 3: Undesirable contribution of one of inner dimensions but acceptable total impact,

Case 4: Undesirable contribution of both inner dimensions and unacceptable total impact.

According to the results of the first and third cases, in general such examinations are three dimensional, assumptions of the qualified exam is almost satisfied in both cases. However, in the second case course has high relation with background knowledge and this is not required situation. In the fourth case actual model and pattern model are very far from each other and that situation is discussed according to two different facts. On the one hand actual examination is highly concentrated on one dimension (new course materials itself) which is the indicator of very week connection with prerequisites; on the other hand relation among the prerequisite examinations and the actual examination is observed very highly, actual examination is still three dimensional but effects of prerequisites are undesirably exceeds expected contributions.

Publications

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